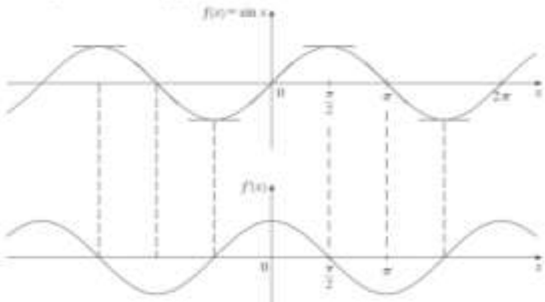
Derivative of Trigonometric Functions

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the defin ition of a derivative, we have



$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]
\end{aligned}
$$

1 $$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when con puting a limit as $h \to 0$, we have

$$\lim_{h\to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h\to 0} \cos x = \cos x$$

The limit of $(\sin h)/h$ is not so obvious. In Example 3 in Section 2.2 we made the guess, on the basis of numerical and graphical evidence, that

2 $$\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = 1$$

$$
\begin{aligned}
\lim_{\theta\to 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta\to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta(\cos \theta + 1)} \\
&= \lim_{\theta\to 0} \frac{-\sin^2\theta}{\theta(\cos \theta + 1)} = -\lim_{\theta\to 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \\
&= -\lim_{\theta\to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin \theta}{\cos \theta + 1} \\
&= -1 \cdot \left(\frac{0}{1+1} \right) = 0 \qquad \text{(by Equation 2)}
\end{aligned}
$$

$$\lim_{\theta\to 0} \frac{\cos \theta - 1}{\theta} = 0$$

If we now put the limits (2) and (3) in (1), we get

$$f'(x) = \lim_{h\to0} \sin x \cdot \lim_{h\to0} \frac{\cos h - 1}{h} + \lim_{h\to0} \cos x \cdot \lim_{h\to0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

Home Work

Using the same methods as in the proof of Formula

5 $$\frac{d}{dx}(\cos x) = -\sin x$$

The tangent function can also be differentiated by using the definition of a derivative,

but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

6 $$\frac{d}{dx}(\tan x) = \sec^2 x$$

In General

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example (1): Derivative of f(x)=sec(x) / (1+tan(x)

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$
$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$
$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$
$$= \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}$$

Example (2): Let $f(x) = \dfrac{3+x}{3-x}$, $x \neq 3$. Evaluate $f'(2)$

$$f'(x) = \frac{(3-x)\dfrac{d}{dx}(3+x) - (3+x)\dfrac{d}{dx}(3-x)}{(3-x)^2} = \frac{(3-x)(1)-(3+x)(-1)}{(3-x)^2} = \frac{6}{(3-x)^2}$$

Putting $x = 2$, we find $f'(2) = 6$.

Example (3): Let $f(x) = \sqrt{2x-1}$. Evaluate $f'(5)$

By using rules of differentiation we find $f'(x) = \frac{d}{dx}(2x-1)^{1/2} = \frac{1}{2}(2x-1)^{-1/2}\frac{d}{dx}(2x-1) = (2x-1)^{-1/2}$. Then $f'(5) = 9^{-1/2} = \frac{1}{3}$.

Example (4):

(*a*) Show directly from definition that the derivative of $f(x) = x^3$ is $3x^2$.

(*a*) $$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}[(x+h)^3 - x^3]$$
$$= \frac{1}{h}[x^3 + 3x^2h + 3xh^2 + h^3] - x^3] = 3x^2 + 3xh + h^2$$

Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3x^2$$

(*b*) Show from definition that $\dfrac{d}{dx}\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

Home Work

EXAMPLE 5 Derivatives Involving the Sine

You Try It

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x.$$

Video

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x \qquad \text{Product Rule}$$
$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}.$$

Example (6):

$y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}.$$

Example (7)

Find y'' if $y = \sec x$.

Solution

$$
\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}
$$

Exercises

1. $y = -10x + 3\cos x$
2. $y = \frac{3}{x} + 5\sin x$
3. $y = \csc x - 4\sqrt{x} + 7$
4. $y = x^2 \cot x - \frac{1}{x^2}$
5. $y = (\sec x + \tan x)(\sec x - \tan x)$
6. $y = (\sin x + \cos x)\sec x$
7. $y = \frac{\cot x}{1 + \cot x}$
8. $y = \frac{\cos x}{1 + \sin x}$
9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
11. $y = x^2 \sin x + 2x\cos x - 2\sin x$
12. $y = x^2 \cos x - 2x\sin x - 2\cos x$