Finding Antiderivatives

> **DEFINITION** **Antiderivative**
> A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = x^2 + \sin x$

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives

$$F(0) = -\cos 0 + C = -1 + C.$$

Since $F(0) = 3$, solving for C gives $C = 4$. So

$$F(x) = -\cos x + 4$$

is the antiderivative satisfying $F(0) = 3$. ■

EXAMPLE 3 Finding Antiderivatives Using Table 4.2

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Solution

(a) $F(x) = \dfrac{x^6}{6} + C$ — Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$ — Formula 1 with $n = -1/2$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

(d) $I(x) = \dfrac{\sin (x/2)}{1/2} + C = 2 \sin \dfrac{x}{2} + C$ — Formula 3 with $k = 1/2$

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE 4 Using the Linearity Rules for Antiderivative

Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

$$F(x) = 3G(x) + H(x) + C$$
$$= 6\sqrt{x} - \frac{1}{2}\cos 2x + C$$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION **Indefinite Integral, Integrand**

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\, dx = x^2 + C,$$

$$\int \cos x\, dx = \sin x + C,$$

$$\int (2x + \cos x)\, dx = x^2 + \sin x + C.$$

EXAMPLE 7 Indefinite Integration Done Term-by-Term and Rewriting the Constant of Integration

You Try It

Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx\\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx\\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3)\\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx\\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. **a.** $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$

2. **a.** $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$

3. **a.** $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$

4. **a.** $2x^{-3}$ **b.** $\dfrac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$

5. **a.** $\dfrac{1}{x^2}$ **b.** $\dfrac{5}{x^2}$ **c.** $2 - \dfrac{5}{x^2}$

6. **a.** $-\dfrac{2}{x^3}$ **b.** $\dfrac{1}{2x^3}$ **c.** $x^3 - \dfrac{1}{x^3}$

7. **a.** $\dfrac{3}{2}\sqrt{x}$ **b.** $\dfrac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

8. **a.** $\dfrac{4}{3}\sqrt[3]{x}$ **b.** $\dfrac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

9. **a.** $\dfrac{2}{3}x^{-1/3}$ **b.** $\dfrac{1}{3}x^{-2/3}$ **c.** $-\dfrac{1}{3}x^{-4/3}$

10. **a.** $\dfrac{1}{2}x^{-1/2}$ **b.** $-\dfrac{1}{2}x^{-3/2}$ **c.** $-\dfrac{3}{2}x^{-5/2}$

11. **a.** $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$

12. **a.** $\pi \cos \pi x$ **b.** $\dfrac{\pi}{2} \cos \dfrac{\pi x}{2}$ **c.** $\cos \dfrac{\pi x}{2} + \pi \cos x$

13. **a.** $\sec^2 x$ **b.** $\dfrac{2}{3} \sec^2 \dfrac{x}{3}$ **c.** $-\sec^2 \dfrac{3x}{2}$

14. **a.** $\csc^2 x$ **b.** $-\dfrac{3}{2} \csc^2 \dfrac{3x}{2}$ **c.** $1 - 8 \csc^2 2x$

15. **a.** $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc \dfrac{\pi x}{2} \cot \dfrac{\pi x}{2}$

16. **a.** $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec \dfrac{\pi x}{2} \tan \dfrac{\pi x}{2}$

Finding Indefinite Integrals

In Exercises 17–54, find the most general antiderivative or indefinite integral. Check your answers by differentiation.

17. $\int (x+1)\,dx$

18. $\int (5-6x)\,dx$

19. $\int \left(3t^2+\frac{t}{2}\right)dt$

20. $\int \left(\frac{t^2}{2}+4t^3\right)dt$

21. $\int (2x^3-5x+7)\,dx$

22. $\int (1-x^2-3x^5)\,dx$

23. $\int \left(\frac{1}{x^2}-x^2-\frac{1}{3}\right)dx$

24. $\int \left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx$

25. $\int x^{-1/3}\,dx$

26. $\int x^{-5/4}\,dx$

27. $\int \left(\sqrt{x}+\sqrt[3]{x}\right)dx$

28. $\int \left(\frac{\sqrt{x}}{2}+\frac{2}{\sqrt{x}}\right)dx$

29. $\int \left(8y-\frac{2}{y^{1/4}}\right)dy$

30. $\int \left(\frac{1}{7}-\frac{1}{y^{5/4}}\right)dy$

31. $\int 2x(1-x^{-3})\,dx$

32. $\int x^{-3}(x+1)\,dx$

33. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt$

34. $\int \frac{4+\sqrt{t}}{t^3}\,dt$

35. $\int (-2\cos t)\,dt$

36. $\int (-5\sin t)\,dt$

37. $\int 7\sin\frac{\theta}{3}\,d\theta$

38. $\int 3\cos 5\theta\,d\theta$

39. $\int (-3\csc^2 x)\,dx$

40. $\int \left(-\frac{\sec^2 x}{3}\right)dx$

41. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

42. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$