> **THEOREM 6 Substitution in Definite Integrals**
> If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then
> $$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

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EXAMPLE 1 Substitution by Two Methods

Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

$\sqrt{x^3+1}$ $\quad 3x^2$

$f(g(x))$

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

الطريقة العكسية → $\displaystyle\int (x^3+1)^{1/2}\cdot 3x^2\,dx$

طريقة بالتعويض

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_0^2 \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx. \text{ When } x=-1,\ u=(-1)^3+1=0. \text{ When } x=1,\ u=(1)^3+1=2.$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2 \qquad \text{Evaluate the new definite integral.}$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

$u = x^3+1$

$du = 3x^2\,dx$

$\int \sqrt{x^3+1}\cdot 3x^2\,dx$

$\int \sqrt{u}\,du = \dfrac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_{-1}^{1}$

$\dfrac{(x^3+1)^{3/2}}{3/2}\Big|_{-1}^{1}$

$\sqrt{2^3} = \sqrt{2^2\cdot 2} = 2\sqrt{2}$

$u = x^3+1$

$u = (-1)^3+1 = -1+1 = 0$

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Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$u = x^3 + 1 = 1^3 + 1 = 1 + 1 = 2$

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3 + 1,\ du = 3x^2\,dx.$$

$$= \boxed{\frac{2}{3}u^{3/2} + C} \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3 + 1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

$\frac{2}{3}u^{3/2}\Big|_0^2$

$\frac{2}{3}(x^3+1)^{3/2}\Big|_{-1}^{1}$

EXAMPLE 2 Using the Substitution Formula

You Try It

طريقة أخرى للحل

$$\int_{\pi/4}^{\pi/2} \cot\theta\,\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du) \qquad \text{Let } u = \cot\theta,\ du = -\csc^2\theta\,d\theta,\ -du = \csc^2\theta\,d\theta.$$

When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

$(-1)\int \cot(-\csc^2\theta)\,d\theta$

$-\frac{\cot^2}{2} + c$

$u = \cot\theta$
$du = -\csc^2\theta\,d\theta \rightarrow -du = \csc^2\theta\,d\theta$

$u = \cot\theta$
$u_1 = \cot\frac{\pi}{4} = 1$
$u_2\ \cot\frac{\pi}{2} = 0$

$\int_1^0 u\,du = -\frac{u^2}{2} = -\left[\frac{u^2}{2}\right]_1^0$

$-\int_a^b \rightarrow \int_b^a$

$-\left[\frac{u^2}{2}\right]_1^0 = \frac{u^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$

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Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

$f(-x) = -f(x)$
$f(x) = x^3$
$f(-x) = (-x)^3 = -(x^3) = -f(x)$

$f(-x) = f(x)$
$f(x) = x^2$
$f(-x) = (-x)^2 = x^2$
$f(-x) = f(x)$

$\int_{-1} \quad \int_{-2} \ldots \int_{-a}^{a}$

$\int_{-1}^{1}\cos x\,dx = 2\int_0^1 \cos x\,dx$

$\therefore f(x) = \cos x$
$\cos(-x) = \cos x$ دالة زوجية

$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$

$\int_{-1}^{1} x\,dx = 0$

$f(x) = x \rightarrow$ هي داله فرديه

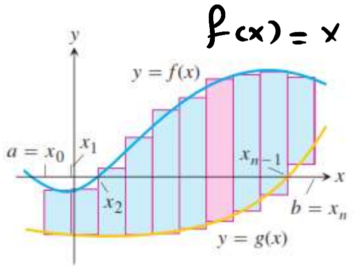


FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.

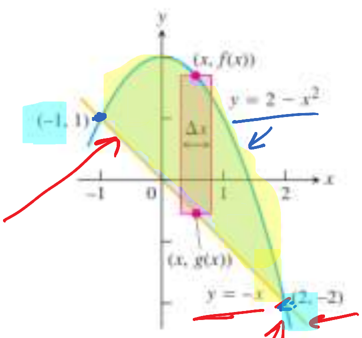
EXAMPLE 3 Integral of an Even Function

Evaluate $\displaystyle\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx.$

$f(-x) = (-x)^4 - 4(-x)^2 + 6 = x^4 - 4x^2 + 6 = f(x)$

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$



$y = -x$

FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Area Between Intersecting Curves

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.30). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$2 - x^2 = -x$	Equate $f(x)$ and $g(x)$.
$x^2 - x - 2 = 0$	Rewrite.
$\boxed{(x + 1)(x - 2) = 0}$	Factor.
$x = -1, \quad x = 2.$	Solve.

$\therefore x + 1 = 0 \rightarrow 1 =$

$x = -1$

$x - 2 = 0$

$x = 2$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

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$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2}[(2 - x^2) - (-x)]\,dx$$
$$= \int_{-1}^{2}(2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

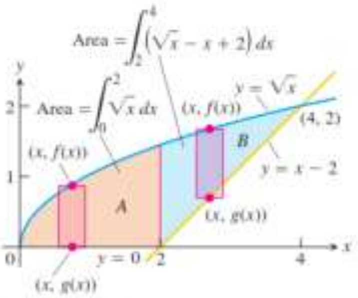

FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

EXAMPLE 5 Changing the Integral to Match a Boundary Change

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.31) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (there is agreement at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.31.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$y_1 = y_2$

$$\sqrt{x} = x - 2 \qquad \text{Equate } f(x) \text{ and } g(x).$$
$$x = (x - 2)^2 = x^2 - 4x + 4 \qquad \text{Square both sides.}$$

$-x$

$-5x$

$$x^2 - 5x + 4 = 0 \qquad \text{Rewrite.}$$
$$(x - 1)(x - 4) = 0 \qquad \text{Factor.}$$
$$x = 1, \quad x = 4. \qquad \text{Solve.}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

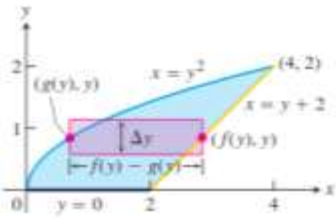

FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

EXAMPLE 6 Find the area of the region in Example 5 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.32). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$y = \sqrt{x} \rightarrow y^2 = x$,

$$y + 2 = y^2 \qquad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$
$$y^2 - y - 2 = 0 \qquad \text{Rewrite.}$$
$$(y + 1)(y - 2) = 0 \qquad \text{Factor.}$$
$$y = -1, \quad y = 2 \qquad \text{Solve.}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$y_1 = \sqrt{x} \Rightarrow y^2 = x$ قوة 2

$y_2 = x - 2 \rightarrow y - 2 = x$ قانون المساحة

$$A = \int_a^b [f(y) - g(y)]\, dy = \int_0^2 [y + 2 - y^2]\, dy$$
$$= \int_0^2 [2 + y - y^2]\, dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$A = \int_a^b [f(y) - g(y)]\, dy = \int_0^2 [y + 2 - y^2]\, dy$$
$$= \int_0^2 [2 + y - y^2]\, dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$

Combining Integrals with Formulas from Geometry

The fastest way to find an area may be to combine calculus and geometry.

EXAMPLE 7 The Area of the Region in Example 5 Found the Fastest Way

Find the area of the region in Example 5.

Solution The area we want is the area between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis, *minus* the area of a triangle with base 2 and height 2 (Figure 5.33):

$$\begin{aligned}
\text{Area} &= \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2) \\
&= \frac{2}{3} x^{3/2} \Big]_0^4 - 2 \\
&= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.
\end{aligned}$$