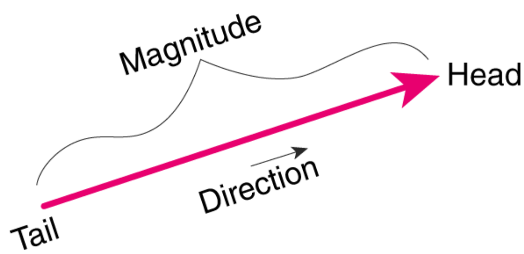
Vectors

Vector is a physical quantity that has both direction and magnitude. In other words, the vectors are defined as an object comprising both magnitude and direction. It describes the movement of the object from one point to another. The below figure shows the vector with head, tail, magnitude and direction.



Types of Vectors List

There are 10 types of vectors in mathematics which are:

1. Zero Vector
2. Unit Vector
3. Position Vector
4. Co-initial Vector
5. Like and Unlike Vectors
6. Co-planar Vector
7. Collinear Vector
8. Equal Vector
9. Displacement Vector
10. Negative of a Vector

All these vectors are extremely important and the concepts are frequently required in mathematics and other higher-level science topics. The detailed explanations on each of these 10 vector types are given below.

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Zero Vector

A zero vector is a vector when the magnitude of the vector is zero and the starting point of the vector coincides with the terminal point.

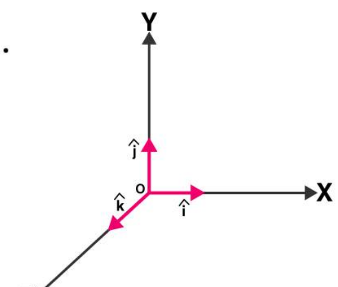
In other words, for a vector $\overrightarrow{AB}$ the coordinates of the point A are the same as that of the point B then the vector is said to be a zero vector and is denoted by 0.
This follows that the magnitude of the zero vector is zero and the direction of such a vector is indeterminate.

Unit Vector

A vector which has a magnitude of unit length is called a unit vector.

Suppose if $\vec{x}$ is a vector having a magnitude x then the unit vector is denoted by $\hat{x}$ in the direction of the vector $\vec{x}$ and has the magnitude equal to 1.

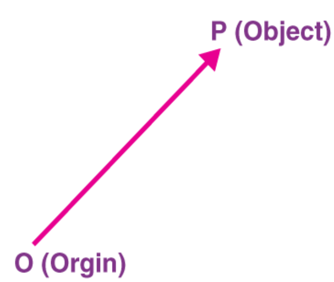
Therefore, $\hat{x} = \frac{\vec{x}}{|x|}$



Unit vector

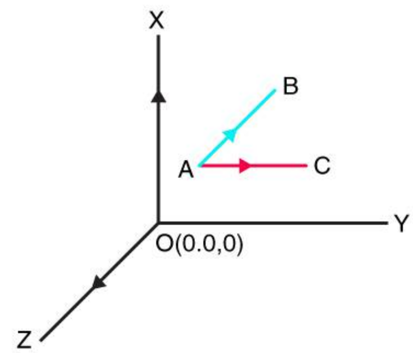
Position Vector

If O is taken as reference origin and P is an arbitrary point in space then the vector $\overrightarrow{OP}$ is called as the position vector of the point.



Co-initial Vectors

The vectors which have the same starting point are called co-initial vectors.



Like and Unlike Vectors

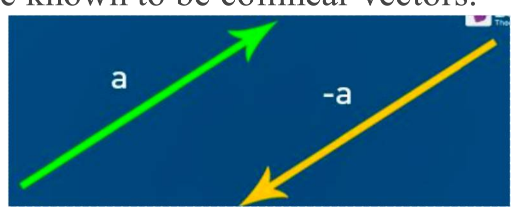
The vectors having the same direction are known as like vectors. On the contrary, the vectors having the opposite direction with respect to each other are termed to be unlike vectors.

Co-planar Vectors

Three or more vectors lying in the same plane or parallel to the same plane are known as co-planar vectors.

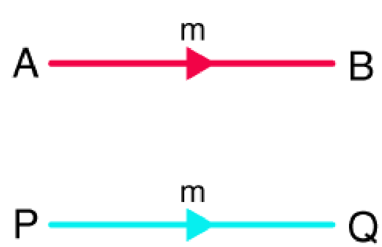
Collinear Vectors

Vectors that lie along the same line or parallel lines are known to be collinear vectors. They are also known as parallel vectors.



Equal Vectors

Two or more vectors are said to be equal when their magnitude is equal and also their direction is the same.



The two vectors shown above, are equal vectors as they have both direction and magnitude equal

Displacement Vector

If a point is displaced from position A to B then the displacement AB represents a vector A which is known as the displacement vector

Negative of a Vector

If two vectors are the same in magnitude but exactly opposite in direction then both the vectors are negative of each other. Assume there are two vectors a and b, such that these vectors are exactly the same in magnitude but opposite in direction then then these vectors can be given as **(a = – b).**

Triangular law of addition

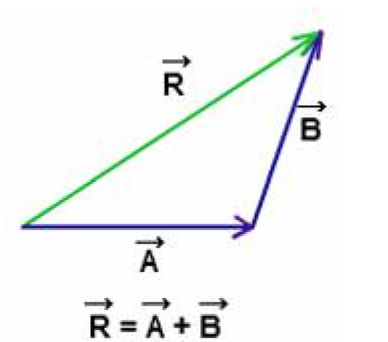
If two forces Vector A and Vector B are acting in the same direction, then its resultant R will be the sum of two vectors.

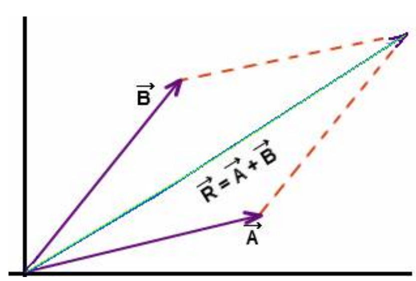
Parallelogram law of addition

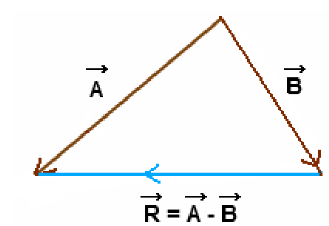
If two forces Vector A and Vector B are represented by the adjacent sides of the parallelogram, then their resultant is represented by the diagonal of a parallelogram drawn from the same point.

Vector Subtraction

If two forces Vector A and Vector B are acting in the direction opposite to each other then their resultant R is represented by the difference between the two vectors.







Dot Product

The dot product (also known as the *scalar product*) of two vectors is defined to be

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta, \tag{1.6}$$

where θ is the angle that $\mathbf{A}$ and $\mathbf{B}$ form when placed tail-to-tail. Since it is a scalar, clearly the product is commutative

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \tag{1.7}$$

Geometrically, $\mathbf{A} \cdot \mathbf{B} = AB_A$ where B_A is the projection of $\mathbf{B}$ on $\mathbf{A}$, as shown in Fig. 1.4. It is also equal to BA_B, where A_B is the projection of $\mathbf{A}$ on $\mathbf{B}$.

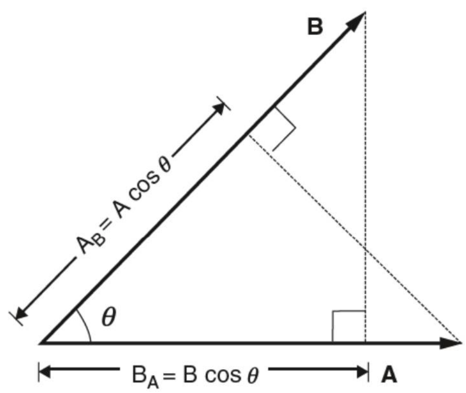


Fig. 1.4. Dot product of two vectors. $\mathbf{A} \cdot \mathbf{B} = AB_A = BA_B = AB\cos\theta$

This shows that the distributive law holds for the dot product

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{1.9}$$

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Example 1.2.1. **Law of cosines.** If A, B, C are the three sides of a triangle, and θ is the interior angle between $\mathbf{A}$ and $\mathbf{B}$, show that

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

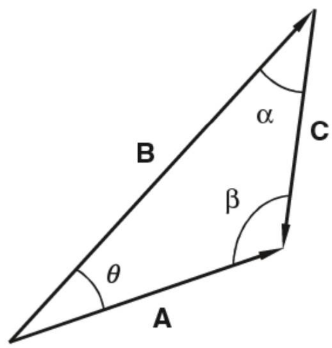

Fig. 1.6. The law of cosine can be readily shown with dot product of vectors, and the law of sine, with cross product

Solution 1.2.1. Let the triangle be formed by the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as shown in Fig. 1.6. Since $\mathbf{C} = \mathbf{A} - \mathbf{B}$,

$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B}.$$

It follows

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

Vector Components

An arbitrary vector **A** can be expanded in terms of these *basis vectors*

where A_x, A_y, and A_z are the projections of $\mathbf{A}$ along the three coordinate axes, they are called components of $\mathbf{A}$.

Since $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular unit vectors, by the definition of dot product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \tag{1.11}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0. \tag{1.12}$$

Because the dot product is distributive, it follows that

$$\begin{aligned}\mathbf{A}\cdot\mathbf{i} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot\mathbf{i} \\ &= A_x\mathbf{i}\cdot\mathbf{i} + A_y\mathbf{j}\cdot\mathbf{i} + A_z\mathbf{k}\cdot\mathbf{i} = A_x,\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{j} = A_y, \qquad \mathbf{A}\cdot\mathbf{k} = A_z,$$

the dot product of **A** with any unit vector is the projection of A along the direction of that unit vector (or the component of **A** along that direction). Thus, (1.10) can be written as

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k}. \tag{1.13}$$

Furthermore, using the distributive law of dot product and (1.11) and (1.12), we have

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})\\ &= A_xB_x + A_yB_y + A_zB_z,\end{aligned} \tag{1.14}$$

and

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2. \tag{1.15}$$

Since $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, the angle between **A** and **B** is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{AB}\right). \tag{1.16}$$

Example 1.2.6. Find the angle between $\mathbf{A} = 3\mathbf{i}+6\mathbf{j}+9\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Solution 1.2.6.

$$A = (3^2 + 6^2 + 9^2)^{1/2} = 3\sqrt{14}; \quad B = \left((-2)^2 + 3^2 + 1^2\right)^{1/2} = \sqrt{14},$$

$$\mathbf{A}\cdot\mathbf{B} = 3\times(-2) + 6\times 3 + 9\times 1 = 21$$

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{21}{3\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2},$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°.$$

Example 1.2.8. If $\mathbf{A} = 3\mathbf{i}+6\mathbf{j}+9\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, find the projection of **A** on **B**.

Solution 1.2.8. The unit vector along **B** is

$$\mathbf{n} = \frac{\mathbf{B}}{B} = \frac{-2\mathbf{i}+3\mathbf{j}+\mathbf{k}}{\sqrt{14}}.$$

The projection of **A** on **B** is then

$$\mathbf{A}\cdot\mathbf{n} = \frac{1}{B}\mathbf{A}\cdot\mathbf{B} = \frac{1}{\sqrt{14}}(3\mathbf{i}+6\mathbf{j}+9\mathbf{k})\cdot(-2\mathbf{i}+3\mathbf{j}+\mathbf{k}) = \frac{21}{\sqrt{14}}.$$

Example 1.2.9. The angles between the vector $\mathbf{A}$ and the three basis vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are, respectively, α, β, and γ. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution 1.2.9. The projections of $\mathbf{A}$ on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are, respectively,

$$A_x = \mathbf{A}\cdot\mathbf{i} = A\cos\alpha; \quad A_y = \mathbf{A}\cdot\mathbf{j} = A\cos\beta; \quad A_z = \mathbf{A}\cdot\mathbf{k} = A\cos\gamma.$$

Thus

$$A_x^2 + A_y^2 + A_z^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma = A^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right).$$

Since $A_x^2 + A_y^2 + A_z^2 = A^2$, therefore

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are often denoted l, m, and n, respectively, and they are called the *direction cosine* of $\mathbf{A}$.

Cross product

The vector cross product written as

$$\mathbf{C} = \mathbf{A}\times\mathbf{B} \tag{1.17}$$

is another particular combination of the two vectors $\mathbf{A}$ and $\mathbf{B}$, which is also very useful. It is defined as a vector (therefore the alternative name: *vector product*) with a magnitude

$$C = AB\sin\theta, \tag{1.18}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and a direction perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ in the sense of the advance of a right-hand screw as it is turned from $\mathbf{A}$ to $\mathbf{B}$. In other words, if the fingers of your right hand point in the direction of the first vector $\mathbf{A}$ and curl around toward the second vector $\mathbf{B}$, then your thumb will indicate the positive direction of $\mathbf{C}$ as shown in Fig. 1.12.