With this choice of direction, we see that cross product is anticommutative

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1.19}$$

It is also clear that if $\mathbf{A}$ and $\mathbf{B}$ are parallel, then $\mathbf{A} \times \mathbf{B} = 0$, since θ is equal to zero.

From this definition, the cross products of the basis vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) can be easily obtained

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \tag{1.20}$$

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}. \end{aligned} \tag{1.21}$$

The following example illustrates the cross product of two nonorthogonal vectors. If $\mathbf{V}$ is a vector in the xz-plane and the angle between $\mathbf{V}$ and $\mathbf{k}$, the unit vector along the z-axis, is θ as shown in Fig. 1.13, then

$$\mathbf{k} \times \mathbf{V} = V \sin\theta \mathbf{j}.$$

Since $|\mathbf{k} \times \mathbf{V}| = |\mathbf{k}|\,|\mathbf{V}| \sin\theta = V \sin\theta$ is equal to the projection of $\mathbf{V}$ on the **xy**-plane, the vector $\mathbf{k} \times \mathbf{V}$ is the result of rotating this projection 90° around the **z** axis.

With this understanding, we can readily demonstrate the distributive law of the cross product

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \tag{1.22}$$

cross product $\mathbf{A} \times \mathbf{B}$ in terms of the components of $\mathbf{A}$ and $\mathbf{B}$:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x\mathbf{i} \times \mathbf{i} + A_xB_y\mathbf{i} \times \mathbf{j} + A_xB_z\mathbf{i} \times \mathbf{k} \\ &\quad + A_yB_x\mathbf{j} \times \mathbf{i} + A_yB_y\mathbf{j} \times \mathbf{j} + A_yB_z\mathbf{j} \times \mathbf{k} \\ &\quad + A_zB_x\mathbf{k} \times \mathbf{i} + A_zB_y\mathbf{k} \times \mathbf{j} + A_zB_z\mathbf{k} \times \mathbf{k} \\ &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}. \end{aligned} \tag{1.23}$$

This cumbersome equation can be more neatly expressed as the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \tag{1.24}$$

The Scalar Triple Product

The scalar triple product, as its name may suggest, results in a scalar as its result. It is a means of combining three vectors via cross product and a dot product. Given the vectors

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$$
$$\mathbf{B} = B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}$$
$$\mathbf{C} = C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}$$

a scalar triple product will involve a dot product and a cross product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

It is necessary to perform the cross product before the dot product when computing a scalar triple product,

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

since $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ one can take the dot product to find that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (A_1)\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - (A_2)\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + (A_3)\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

which is simply

Important Formula 3.1.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Formula 3.1.

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Example 3.1.1. Given,

$$\begin{aligned} \mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{B} &= -\,\mathbf{i} + \mathbf{j} \\ \mathbf{C} &= 2\,\mathbf{i} + 2\,\mathbf{j} \end{aligned}$$

Find

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

Solution:
Method 1:

Begin by finding

$$\begin{aligned} \mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= \big((1)(0) - (0)(2)\big)\mathbf{i} - \big((-1)(0) - (0)(2)\big)\mathbf{j} + \big((-1)(2) - (1)(2)\big)\mathbf{k} \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} - 4\,\mathbf{k}. \end{aligned}$$

. . . example continued

Take the dot product with **A** to find

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= (2)(0) + (3)(0) + (-1)(-4) \\ &= 4 \end{aligned}$$

Method 2:
Evaluate the determinant

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= (2)\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - (3)\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= (2)\big((1)(0) - (0)(0)\big) - (3)\big((-1)(0) - (0)(2)\big) + (-1)\big((-1)(2) - (1)(2)\big) \\ &= 4 \end{aligned}$$

Example 3.1.2. Prove that

Important Formula 3.2.

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

Solution:
Notice that there are no brackets given here as the only way to evaluate the scalar triple products is to perform the cross products before performing the dot products[a]. Let

$$\mathbf{A}=A_1\,\mathbf{i}+A_2\,\mathbf{j}+A_3\,\mathbf{k}$$
$$\mathbf{B}=B_1\,\mathbf{i}+B_2\,\mathbf{j}+B_3\,\mathbf{k}$$
$$\mathbf{C}=C_1\,\mathbf{i}+C_2\,\mathbf{j}+C_3\,\mathbf{k}$$

now,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=-\begin{vmatrix}C_1 & C_2 & C_3\\ B_1 & B_2 & B_3\\ A_1 & A_2 & A_3\end{vmatrix}=\begin{vmatrix}C_1 & C_2 & C_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

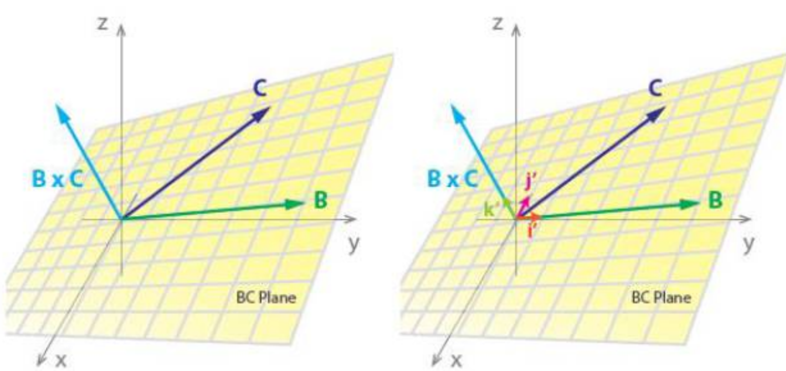
[a] This is due to the fact that if the dot product is evaluate first one would be left with a cross product between a scalar and a vector which is not defined.

The Vector Triple Product

The vector triple product, as its name suggests, produces a vector. It is the result of taking the cross product of one vector with the cross product of two other vectors.

Important Formula 3.3 (Vector Triple Product).

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$



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The Gradient of a scalar function

The vector in the parenthesis is called the *gradient* of φ, and is usually written as grad φ or $\nabla\varphi$,

$$\nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}. \tag{2.61}$$

Since φ is an arbitrary scalar function, it is convenient to define the differential operation in terms of the *gradient operator* ∇ (sometimes known as del or del operator)

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{2.62}$$

This is a vector operator and obeys the same convention as the derivative notation. If a function is placed on the left-hand side of it, $\varphi\nabla$ is still an operator and by itself means nothing. What is to be differentiated must be placed on the right of ∇. When it operates on a scalar function, it turns $\nabla\varphi$ into a vector with definite magnitude and direction. It also has a definite physical meaning.

Example 2.4.1. Show that $\nabla r = \hat{\mathbf{r}}$ and $\nabla f(r) = \hat{\mathbf{r}}\mathrm{d}f/\mathrm{d}r$, where $\hat{\mathbf{r}}$ is a unit vector along the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and r is the magnitude of $\mathbf{r}$.

Solution 2.4.1.

$$\nabla r = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) r,$$

$$\mathbf{i}\frac{\partial r}{\partial x} = \mathbf{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} = \frac{\mathbf{i}x}{(x^2+y^2+z^2)^{1/2}} = \frac{\mathbf{i}x}{r}, \quad \text{etc.}$$

$$\nabla r = \frac{\mathbf{i}x}{r} + \frac{\mathbf{j}y}{r} + \frac{\mathbf{k}z}{r} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.$$

$$\nabla f(r) = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z},$$

$$\mathbf{i}\frac{\partial f}{\partial x} = \mathbf{i}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x} = \mathbf{i}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r}, \quad \text{etc.}$$

$$\nabla f(r) = \mathbf{i}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r} + \mathbf{j}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{y}{r} + \mathbf{k}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{z}{r} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = \hat{\mathbf{r}}\frac{\mathrm{d}f}{\mathrm{d}r}.$$

Example 2.4.2. Show that $(\mathbf{A}\cdot\boldsymbol{\nabla})\,\mathbf{r}=\mathbf{A}$.

Solution 2.4.2.

$$\begin{aligned}(\mathbf{A}\cdot\boldsymbol{\nabla})\,\mathbf{r} &= \left[(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\right]\mathbf{r}\\ &= \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\\ &= A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}=\mathbf{A}.\end{aligned}$$

Example 2.4.4. Find the maximum rate of increase for the surface $\varphi(x,y,z)=100+xyz$ at the point $(1,3,2)$. In which direction is the maximum rate of increase?

Solution 2.4.4. The maximum rate of increase is $|\boldsymbol{\nabla}\varphi|_{1,3,2}$.

$$\boldsymbol{\nabla}\varphi=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(100+xyz)=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k},$$

$$|\boldsymbol{\nabla}\varphi|_{1,3,2}=|6\mathbf{i}+2\mathbf{j}+3\mathbf{k}|=(36+4+9)^{1/2}=9.$$

The direction of the maximum increase is given by

$$\boldsymbol{\nabla}\varphi|_{1,3,2}=6\mathbf{i}+2\mathbf{j}+3\mathbf{k}.$$

Example 2.4.5. Find the rate of increase for the surface $\varphi(x,y,z)=xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

Solution 2.4.5.

$$\boldsymbol{\nabla}\varphi=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(xy^2+yz^3\right)=y^2\mathbf{i}+\left(2xy+z^3\right)\mathbf{j}+3yz^2\mathbf{k},$$

$$\boldsymbol{\nabla}\varphi_{2,-1,1}=\mathbf{i}-3\mathbf{j}-3\mathbf{k}.$$

The unit vector along $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is

$$\mathbf{n}=\frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1+4+4}}=\frac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k}).$$

The rate of increase is

$$\frac{d\varphi}{dr}=\boldsymbol{\nabla}\varphi\cdot\mathbf{n}=(\mathbf{i}-3\mathbf{j}-3\mathbf{k})\cdot\frac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})=-\frac{11}{3}.$$

Example 2.4.6. Find the equation of the tangent plane to the surface described by $\varphi(x,y,z)=2xz^2-3xy-4x=7$ at the point $(1,-1,2)$.

Solution 2.4.6. If $\mathbf{r}_0$ is a vector from the origin to the point $(1,-1,2)$ and $\mathbf{r}$ is a vector to any point in the tangent plane, then $\mathbf{r}-\mathbf{r}_0$ lies in the tangent plane. The tangent plane at $(1,-1,2)$ is normal to the gradient at that point, so we have

$$\boldsymbol{\nabla}\varphi|_{1,-1,2}\cdot(\mathbf{r}-\mathbf{r}_0)=0.$$

$$\boldsymbol{\nabla}\varphi|_{1,-1,2}=\left[\left(2z^2-3y-4\right)\mathbf{i}-3x\mathbf{j}-4xz\mathbf{k}\right]_{1,-1,2}=7\mathbf{i}-3\mathbf{j}+8\mathbf{k}.$$

Therefore the tangent plane is given by the equation

$$(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})\cdot[(x-1)\,\mathbf{i}+(y+1)\,\mathbf{j}+(z-2)\,\mathbf{k}]=0,$$
$$7(x-1)-3(y+1)+8(z-2)=0,$$
$$7x-3y+8z=26.$$

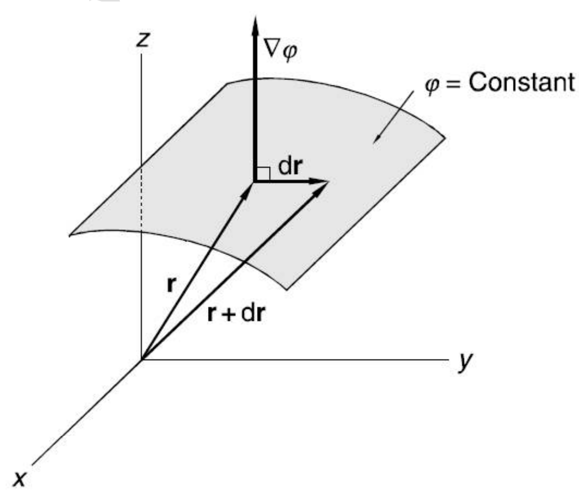

Fig. 2.6. Gradient of a scalar function. $\boldsymbol{\nabla}\varphi$ is a vector normal to the surface of $\varphi=$ constant

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The Divergent of a Vector

Just as we can operate with ∇ on a scalar field, we can also operate with ∇ on a vector field $\mathbf{A}$ by taking the dot product. With their components, this operation gives

$$\nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \quad (2.69)$$

Just as the dot product of two vectors is a scalar, $\nabla \cdot \mathbf{A}$ is also a scalar. This sum, called the *divergence* of $\mathbf{A}$ (or div $\mathbf{A}$), is a special combination of derivatives.

Example 2.5.1. Show that $\nabla \cdot \mathbf{r} = 3$ and $\nabla \cdot \mathbf{r} f(r) = 3f(r) + r(\mathrm{d}f/\mathrm{d}r)$.

Solution 2.5.1.

$$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$
$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$$\nabla \cdot \mathbf{r} f(r) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}xf(r) + \mathbf{j}yf(r) + \mathbf{k}zf(r))$$
$$= \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)]$$
$$= f(r) + x\frac{\partial f}{\partial x} + f(r) + y\frac{\partial f}{\partial y} + f(r) + z\frac{\partial f}{\partial z}$$
$$= 3f(r) + x\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x} + y\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial y} + z\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial z}.$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{1/2} = \frac{1}{2}\frac{2x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r};$$
$$\frac{\partial r}{\partial y} = \frac{y}{r}; \qquad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

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$$\nabla \cdot \mathbf{r} f(r) = 3f(r) + \frac{x^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{y^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{z^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}$$

$$= 3f(r) + \frac{x^2+y^2+z^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = 3f(r) + r\frac{\mathrm{d}f}{\mathrm{d}r}.$$